



The Open University

MU120
Open Mathematics

Resource Book B

Units 6–9





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Contents

Unit 6	Maps	4
Unit 7	Graphs	7
Unit 8	Symbols	11
Unit 9	Music	15
Solutions		20
Unit 6		20
Unit 7		23
Unit 8		27
Unit 9		30

Unit 6 Maps

Question 1

Unit 6, Sec. 2.1 and 2.2

Use your Ordnance Survey (OS) map extract of the Peak District to do the following.

- (a) Identify the feature located at each of the grid references given here:
168829 147844 136880 150829
- (b) Find the grid references of the locations listed below:
top of Crook Hill, north of Ladybower reservoir;
Druid's Stone, about 3 km north of Hollins Cross;
Blue John Cavern, about 2 km west of Castleton;
railway station about 1 km east of Hope.

Question 2

Unit 6, Sec. 2.1 and 2.2

The Ordnance Survey's National Grid divides Great Britain into squares 100 km by 100 km. What size would a map of one of these squares be if it were drawn to a scale of 1:50 000?

Question 3

Unit 6, Sec. 2.2, 4.2 and 5

- (a) On your OS map extract, measure the straight-line distances from the cairn at grid reference 147874 to the features listed in the table below, and then complete the table. Give the map distances to the nearest millimetre and the ground distances to the nearest 100 metres.

Feature	Grid reference	Map distance/ cm	Ground distance/ km
Guide post	160873		
Cairn	149882		
Madwoman's Stones	136880		
Shooting cabin	142887		

- (b) Instead of measuring the straight-line distances on the map, it is possible to calculate the ground distances from the grid references by using Pythagoras' theorem (see the diagram in Section 6.2 of the *Calculator Book*). Check your answers to part (a) by using this method.

Question 4

Unit 6, Sec. 3.2

In 1994, magnetic north was 5° west of grid north in the region covered by your OS map extract, and was estimated to be decreasing by about $\frac{1}{2}^\circ$ every 3 years. Use this information to predict the direction of magnetic north relative to grid north in that region in 2004.

Question 5*Unit 6, Sec. 3.2*

To check her position, a walker takes two compass bearings. The first on the chimney at grid reference 164824 (on your OS map extract) comes out at 119° . The second on Win Hill (186850) is 83° . Where is she (assuming it is the year 1994 when, for the region covered by this map extract, magnetic north was 5° west of grid north)?

Question 6*Unit 6, Sec. 3.3 and Calc. Book, Sec. 6.2*

A walk is planned to start and finish in the car park (122832) below Mam Tor. The route climbs to Mam Tor and then follows the path to Hollins Cross (136845). From there, the path turns north and descends to Backtor Bridge to join the road at 136855. It then follows the road to just past the T-junction where it turns left (123852) to follow the path up Harden Clough. The path meets the road from Barber Booth at 123837, and then follows the road back to the car park.

Use your OS map extract and Naismith's rule to estimate how long the walk will take.

Question 7*Unit 6, Sec. 4.1*

Using your OS map extract, estimate the change in height relative to the cairn at grid reference 147874, for the features listed in the table below. Use your answers to Question 3 to estimate the distance from the cairn to each feature, and the average gradient of the straight line from the cairn to the feature.

For a definition of 'gradient', see *Unit 6*, p.76.

Feature	Grid reference	Height/m	Change in height/m	Distance/m	Average gradient
Guide post	160873				
Cairn	149882				
Madwoman's Stones	136880				
Shooting cabin	142887				

Question 8*Unit 6, Sec. 2.2 and Sec. 4.4*

An old UK road atlas is drawn to a scale of 3 miles to 1 inch. What is the map scale expressed as a ratio? What area on the ground in hectares does 1 square inch in the atlas represent?

In the imperial system of units, there are 12 inches to 1 foot, 3 feet to 1 yard and 1760 yards to 1 mile. 1 mile is equivalent to approximately 1.61 km.

Question 9*Unit 6, Sec. 4.4*

- An area of 50 ha on the ground is represented on a map by an area of 1 cm^2 . What is the scale of the map?
- If the map scale were $1 : 100\,000$, what area in hectares would 1 cm^2 represent?

Question 10

Calc. Book, Sec. 6.1

Use your calculator to plot a line graph representing the profile of the straight-line route between the cairns at grid references 147874 and 149882. Experiment to see the effect of different window settings on the shape of your graph. At what distance from the cairn at 147874 does the gradient become zero?

Question 11

Calc. Book, Sec. 6.1 and 6.2

A walker plans the following walk:

Location	Grid reference
Yorkshire Bridge	197849
West bank of Ladybower Reservoir	197856
Near New Barn	190860
Saddle point on Win Hill	183850
Win Hill Pike	187851
Yorkshire Bridge	197849

- Enter appropriate eastings and northings on your calculator, and plot a line graph representing the route of the walk.
- Using Pythagoras' theorem, carry out appropriate list arithmetic to estimate the distance of each section of the walk, and then estimate the total distance of the walk.

Unit 7 Graphs

Question 1

Unit 7, Sec. 1.2

At the exchange rate in early 2002, £300 sterling was equivalent to 470 euros.

- (a) Using these data and the fact that £0 sterling is equivalent to 0 euros, draw a graph to convert between pounds sterling (on the horizontal axis) and euros (on the vertical axis). Your axes should extend to £700 and 1000 euros.
- (b) From your graph, convert £400 into euros, and 750 euros into pounds.
- (c) What type of relationship does your graph show?

Question 2

Unit 7, Sec. 1.2

Estimates of the bird populations of two bird sanctuaries are compared to see which sanctuary will have the larger population in the future. The bird population of sanctuary A is now about 5200 and is increasing steadily at an average rate of 500 birds per year. That of sanctuary B is currently 6000 and is increasing steadily at an average rate of 300 birds per year.

- (a) Write down the word formulas that can be used to predict the bird populations of the two sanctuaries in future years.
- (b) Draw graphs showing the estimates of the two bird populations over a six-year period from the present. When do these estimates predict that the populations of the two sanctuaries will be equal?

Question 3

Unit 7, Sec. 1.3

Complete the table below for the relationship where the y -coordinate is equal to the square of the x -coordinate ($y = x^2$).

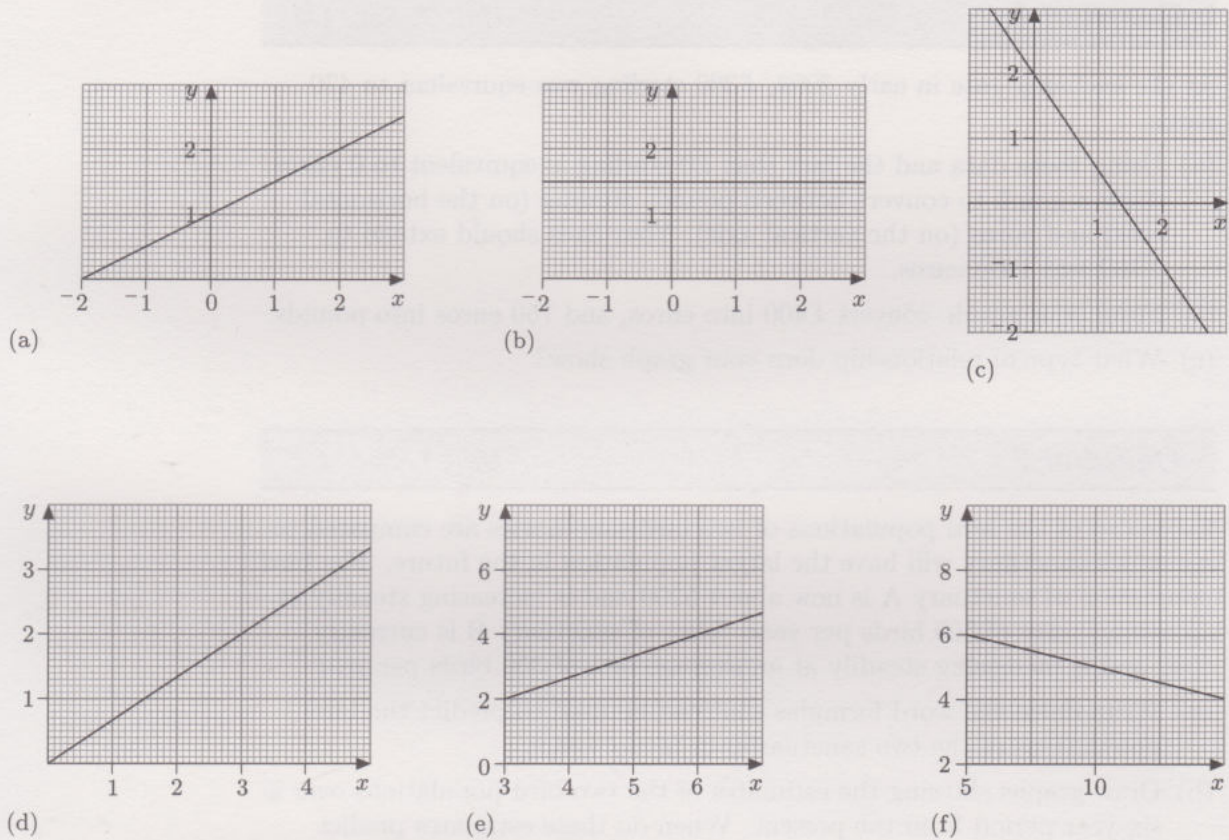
x -coordinate	y -coordinate	Coordinate pair
-2.5	6.25	(-2.5, 6.25)
-1.5	2.25	(-1.5, 2.25)
-0.5		
0		
1		
2		
3		

Using your completed table, make a plot of this relationship on graph paper. How does the gradient of the graph change as the value of the x -coordinate increases from -2.5 to 3?

Question 4

Unit 7, Sec. 1.2 and 1.3

Find the intercept and the gradient of each of the following straight-line graphs.



Question 5

Unit 7, Sec. 2.1

A cross-channel ferry usually takes 2 hours to make the 40 km crossing from England to France. Assuming that the ferry leaves at 12 noon (all the times used are in GMT) and the speed of the ferry is constant during the crossing, draw a distance–time graph to represent the journey. From your graph find:

- when the ferry is 15 km from England, and when it is 35 km from England;
- where the ferry is at 12.15 p.m., and at 1.10 p.m.;
- the gradient of the graph;
- the average speed of the ferry during the crossing.

Question 6

Unit 7, Sec. 2.1

A ship that was travelling at about 10 knots (nautical miles per hour) directly towards a port radioed for a tug when it was 10 nautical miles away from the port. The tug set off from the port 30 minutes later, travelling towards the ship at about 5 knots. Draw the distance–time graphs for the ship and the tug on the same axes, and hence predict where and when the vessels will meet. When should the tug start looking for the ship?

Question 7*Calc. Book, Sec. 7.1*

- (a) With the data given in the answer to Question 6 for *Unit 6* (at the back of this Resource Book), use your calculator to draw a distance–time graph for the planned walk around Mam Tor.
- (b) From the graph, estimate for which legs of the walk the fastest and slowest speeds are predicted.
- (c) Calculate the average speed for each section of the walk.
- (d) What is the average speed for the entire walk?

Question 8*Calc. Book, Sec. 7.2*

Some British recipe books give capacities in fluid ounces (fl. oz), while others use millilitres (ml). According to a dictionary, 20 fl. oz (or 1 pint) are equivalent to 0.568 litre.

- (a) Calculate the number of millilitres equivalent to 1 fl. oz.
- (b) Use your calculator to display a graph that converts fluid ounces into millilitres. Choose window settings such that capacities up to 1 pint can be represented.
- (c) Use TABLE to estimate:
 - (i) the equivalent of 8 fl. oz in millilitres;
 - (ii) the equivalent of 200 ml in fluid ounces.
- (d) Repeat these estimates using the TRACE facility.

Question 9*Unit 7, Sec. 2 and Calc. Book, Sec. 7.2*

You are planning a coach journey across France, and expect to average 95 km per hour.

- (a) Write down the formula relating the distance travelled in kilometres to the time taken in hours, in a form suitable for your calculator.
- (b) Store the formula in your calculator, and use it to produce a table giving your expected distance every half-hour for $4\frac{1}{2}$ hours.
- (c) Suppose that you want to have a short stop after about two hours. How far would you have gone after travelling for two hours?
- (d) Suppose that there is not a convenient service area after two hours, but there is one at 175 km and another at 220 km. Use the stored formula on your calculator to estimate how long you would have been travelling when you pass each of these service areas.

Question 10

Calc. Book, Sec. 7.2

If a mass of P lb is equivalent to K kg, then P and K are related as follows:

$$K = 0.45P.$$

This is the formula for converting mass in pounds to mass in kilograms.

Write down a similar formula for converting mass in pounds to mass in stones (a stone is 14 lb). Use your calculator to produce a table corresponding to these formulas, which will allow you to convert directly from stones and half stones to kilograms.

Unit 8 Symbols

Question 1

Prep. Res. Book A and Unit 8, Sec. 1.1

Simplify the following expressions, which involve brackets and positive and negative quantities:

- | | | |
|---|---------------------------------------|---------------------------------|
| (a) $4 - 3 + 5$ | (b) $5 + 8 - 15$ | (c) $-7a + 14a - 3a$ |
| (d) $4 - (-3)$ | (e) $6 + (-8)$ | (f) $12 - (+4)$ |
| (g) $121p + 43p - (-46p)$ | (h) 8×-7 | (i) $3 \times (+4) \times (-9)$ |
| (j) $-4 \times -6 \times -8 \times -12$ | (k) $-2 \times 4 \times -6 \times -3$ | (l) $-4 \div 8$ |
| (m) $\frac{-16q}{-4}$ | (n) $\frac{28}{-7} + (-4 \times -3)$ | (o) $-3(5 - 8)$ |

Question 2

Unit 8, Sec. 1.2

- (a) In a greengrocer's shop, bananas are priced at 95p per kg. Find a formula that can be used to calculate the cost of a purchase of any number of kilograms of bananas.
- (b) Customers are also charged 7p for a carrier bag. Amend your formula to include the cost of a carrier bag. Then convert this formula to give the total cost of the purchase in £s.
- (c) Using the formula obtained in part (b), find the total cost, in £s, of 2.6 kg of bananas in a carrier bag.

Question 3

Unit 8, Sec. 1.2

The cost of local telephone calls is 2.4 p a unit. There is a standing charge of £15.30 a quarter. For a household that makes only local calls, find a formula relating the quarterly bill, in £s, to the number of units used.

Question 4

Unit 8, Sec. 1.2

The dimensions of Noah's ark are given in Genesis 6, 14–15 as follows: 'Make thee an ark of gopher wood ... the length of the ark shall be three hundred cubits, the breadth of it fifty cubits, and the height of it thirty cubits.'

A cubit is usually taken to be about 0.46 m, so the number of cubits, C , is related to the number of metres, M , by the formula $M = 0.46C$. Use this formula to calculate the dimensions of the ark in metres.

Question 5

Unit 8, Sec. 2.2

If $p = 3q + 2$, find the value of p when

- (a) $q = 2$ (b) $q = 1.74$ (c) $q = -1.5$.

Question 6

Unit 8, Sec. 2.2

- (a) What is the value of $2x^2 - 3$ when $x = -2$?
 (b) What is the value of $\frac{2}{3}(4 + 3q)$ when $q = \frac{4}{5}$?

Question 7

Unit 8, Sec. 2.2

- (a) Express the following without $\times$ signs:

$$5 \times N, \quad 5 \times 6 \times z, \quad 20y \times 2.$$

- (b) Simplify the following as far as possible:

$$2y - 7y, \quad 2 \times 5x + 4 \times 3x, \quad 6p + 7p^2 - 4p + 2p^2.$$

Question 8

Unit 8, Sec. 2.3

Simplify these expressions as far as possible:

- (a) $7y + 4x - 3y$ (b) $p + 2r - q + 4p + q$
 (c) $17\clubsuit - 9\spadesuit + 23\spadesuit - 5\clubsuit$ (d) 4 times k minus 13 times k plus s
 (e) $2(A)^2 + (3A)A$ (f) $3ab + 3ab^2 - 2ba + b(ab)$.

Question 9

Unit 8, Sec. 2.5

Remove all the brackets from the following expressions:

- (a) $2(3x - 1)$ (b) $4(k + 5)$ (c) $p(2q + p)$ (d) $2m(a - b + c)$
 (e) $2b(b + b^2 - b^3)$ (f) $-3(x + 2y - z)$ (g) $\frac{1}{3}(6r - 9r^2)$.

Question 10

Unit 8, Sec. 2.5

Multiply out the brackets in these expressions:

- (a) $(x + 2)(x + 5)$ (b) $(2x - 1)(3x + 2)$
 (c) $(3 - y)(4 + y)$ (d) $(5f - 3)(2f - 1)$
 (e) $(2 + x)(5 - y)$ (f) $(p + q)(x + y)$.

Question 11

Unit 8, Sec. 2

Which of the following are equivalent to $16p^2qr^3$?

- (a) $4 \times 4p^2qr^3$ (b) $16r^3qp^2$ (c) $16pq^2r^3$ (d) $16p^3q^2r$
 (e) $2 \times 2 \times 4 \times q \times p \times p \times r \times r \times r$ (f) $\frac{32p^2q^2r^3}{2q}$.

Question 12*Calc. Book, Sec. 8.2*

Press the 'Y=' key on your calculator and input the function $y = 4x - 2.3$.

- Use the method outlined in Section 8.2 of the *Calculator Book* to obtain y values when x takes the values 3, 13.72 and -101.5 .
- Plot the corresponding graph, and use TRACE and ZOOM to check your answers to part (a).

Question 13*Unit 8, Sec. 4.1*

Suppose you want to rearrange the formula $p = 4.2q - 1.7$ to make q the subject.

- What are the two doing steps that have been performed on q to get $4.2q - 1.7$?
- To rearrange the formula to get q in terms of p , you need to undo the steps identified in part (a). Write down the two steps that you would need to perform on $p = 4.2q - 1.7$ to obtain q . Be careful about the order of the steps.
- Apply the steps specified in part (b) to both sides of the formula, and so obtain the formula in the form $q = \dots$

Question 14*Unit 8, Sec. 4.1*

Use the method of Question 13 to make x the subject of the following equations:

- (a) $y = 5 + 12x$ (b) $y = x/3 + 9$ (c) $y = 12 - 2x$ (d) $y = 8x^3$.

Question 15*Unit 8, Sec. 4.2 and 4.3*

- What operations have been performed on x , and in what order, to obtain $2x + 5$?
- By undoing these operations, solve the equation $2x + 5 = 8$.
- Substitute your value of x into the original equation to see if that value satisfies the equation. If it does, your answer is correct.

Question 16*Unit 8, Sec. 4.3*

Solve the following equations. In each case, substitute your answer back into the equation as a check.

- (a) $3x - 2 = 2.5$ (b) $\frac{2}{3}x + 3 = 2$
 (c) $2x + 3 = 21 - 2x$ (d) $5(x - 2) = 8$.

Question 17

Unit 8, Sec. 4.3

Hansel is six years older than Gretel. In two years time he will be three times as old as Gretel. How old is Gretel now?

Question 18

Calc. Book, Sec. 8.4 and 8.5

On your calculator, display the graph of $y = x^3 + 3x^2 - 9x + 3$. Use any of the methods described in the *Calculator Book* to find

- (a) the value of y when $x = 2.25$;
- (b) the values of x when $y = 10$.

Question 19

Calc. Book, Sec. 8.4 and 8.5

An amateur gardener bought 17 rooted cuttings of fuschias via the Internet. Postage and packing cost £2.36, and the total bill was £11.71. Later, he wondered how much each cutting cost.

- (a) Using f to stand for the price of one fuschia in pounds, write down an equation to represent the transaction.
- (b) On your calculator, enter the functions $y = 17x + 2.36$ and $y = 11.71$. Choose suitable WINDOW settings, and plot the graphs.
- (c) Use TRACE and ZOOM to find the x value (correct to 2 decimal places) of the point where the graphs cross.
- (d) What is the connection between the equation you obtained in part (a) and the functions used in part (b)? Hence interpret your solution from part (c).
- (e) Solve the equation from part (a) algebraically. Comment on the accuracy of the two methods that have been used to solve the equation: the algebraic method and the graphical method.

Question 20

Calc. Book, Sec. 8.4

A firework, resting on the ground, is propelled vertically into the air so that its height, h , in metres after t seconds is given by

$$h = 5t - t^2.$$

Display the graph of this function on your calculator. (Remember that you will have to change h to y , and t to x .)

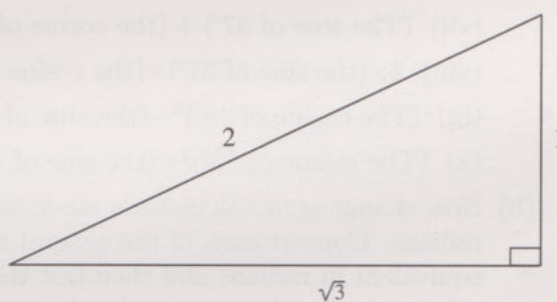
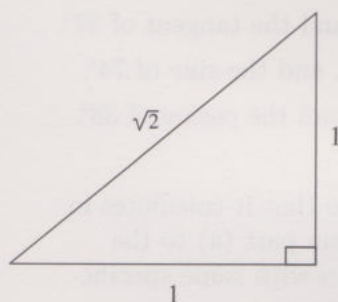
- (a) From the graph, find the maximum height reached by the firework, and the time at which it attains that height. Zoom in until your answers are correct to 2 decimal places.
- (b) If you wished to use your graph to find the times at which the firework is at a height of 6 m, you would need to enter another function on your calculator. What is that function?
- (c) Enter this function, and obtain its graph on the graphing screen. Hence find the times when the firework is at a height of 6 m.

Unit 9 Music

Question 1

Calc. Book, Sec. 9.1

- (a) Use your calculator to evaluate the angles specified below, giving the answers in degrees. Then use the fact that $180^\circ = \pi$ radians in order to obtain the answers in radians (as multiples of π).
- The angle whose sine is $1/2$.
 - The angle whose cosine is $1/2$.
 - The angle whose sine is $(1 \div \sqrt{2})$.
 - The angle whose cosine is $(1 \div \sqrt{2})$.
 - The angle whose sine is $(\sqrt{3} \div 2)$.
 - The angle whose cosine is $(\sqrt{3} \div 2)$.
 - The angle whose tangent is 1.
 - The angle whose tangent is $(1 \div \sqrt{3})$.
 - The angle whose tangent is $\sqrt{3}$.
- (b) Can you identify each of the angles from part (a) in these right-angled triangles?



Question 2

Calc. Book, Sec. 9.1

With your calculator in radian mode, enter $Y1 = \sin(X)$ and $Y2 = \cos(X)$. Then enter appropriate values in TBLSET so as to complete the table below.

Angle (X)/radians	Sine X	Cosine X
0		
$\pi/6$		
$\pi/3$		
$\pi/2$		
$2\pi/3$		
$5\pi/6$		
π		

In Question 1 above, what fractional expression was used instead of 0.86603?

Question 3

Calc. Book, Sec. 9.1

- (a) Use your calculator to evaluate the trigonometric functions listed below, giving your answers to 3 decimal places. You may wish to set the calculator's mode to show 3 decimal places and to calculate in degrees.

Each pair of calculations has been chosen to illustrate some particular general feature of trigonometric functions. In each case, try some similar calculations and then state the general rule. The first one has been done for you.

- (i) The sine of 99° and the sine of 81° :

$$\sin 99^\circ = 0.988 \text{ and } \sin 81^\circ = 0.988.$$

It looks as though angles that are the same amount above and below 90° have the same sine. (Recall that the sine curve is symmetrical about 90° .)

Therefore, in general, $\sin(90^\circ + x) = \sin(90^\circ - x)$.

- (ii) The sine of 183° and the sine of 177° .
 (iii) The cosine of 10° and the cosine of -10° .
 (iv) The cosine of 9° and the cosine of 351° .
 (v) The sine of 3° and the cosine of 87° .
 (vi) The tangent of 5° and the tangent of 185° .
 (vii) $(\text{The sine of } 37^\circ) \div (\text{the cosine of } 37^\circ)$, and the tangent of 37° .
 (viii) $2 \times (\text{the sine of } 37^\circ) \times (\text{the cosine of } 37^\circ)$, and the sine of 74° .
 (ix) $(\text{The cosine of } 19^\circ)^2 - (\text{the sine of } 19^\circ)^2$, and the cosine of 38° .
 (x) $(\text{The cosine of } 19^\circ)^2 + (\text{the sine of } 19^\circ)^2$.

- (b) Now change your calculator's mode setting so that it calculates in radians. Convert each of the general rules from part (a) to the equivalent in radians and then test these rules with some specific values. Again, the first one has been done for you.

- (i) Since $90^\circ = \pi/2$ radians,

$$\sin(90^\circ + x) = \sin(90^\circ - x), \text{ with } x \text{ in degrees,}$$

becomes

$$\sin(\pi/2 + x) = \sin(\pi/2 - x), \text{ with } x \text{ in radians.}$$

$$\text{Check: } \sin(\pi/2 + 0.1) = 0.995 \text{ and } \sin(\pi/2 - 0.1) = 0.995.$$

Question 4

Calc. Book, Sec. 9.1 and 9.2

With your calculator in radian mode, plot the following pairs of graphs. In each case, decide what the graphs have in common and how they differ.

- (a) $Y1 = \cos(X)$ and $Y2 = \cos(2X)$.
 (b) $Y1 = \sin(X) \cos(X)$ and $Y2 = \sin(2X)$.
 (c) $Y1 = \sin(X)/\cos(X)$ and $Y2 = \tan(X)$.
 (d) $Y1 = \cos(X + \pi/2)$ and $Y2 = \sin(X)$.
 (e) $Y1 = (\sin(X))^2$ and $Y2 = (\cos(X))^2$.
 (f) Enter $Y3 = Y1 + Y2$, and repeat part (e).

Question 5**Calc. Book, Sec. 9.3 and Unit 9, Sec. 2**

In days gone by, some guitar makers used the following 'Rule of 18' to determine the positions of the frets:

- Divide the length of the open string (that is, between Fret 0 and the bridge) by 18, and mark off the position of Fret 1 at this distance from Fret 0.
 - Divide the length of the remaining section of string (that is, between Fret 1 and the bridge) by 18, and mark off the position of Fret 2 at this distance from Fret 1.
 - Continue in this way until Fret 20 has been positioned.
- (a) What is the string-length ratio that is used to reduce the effective length of the string (that is, the length between the previous fret and the bridge) at each stage of this process? Store this value in memory location E of your calculator. Compare this with the theoretical string-length ratio needed to position the frets in order to produce an equally-tempered chromatic scale; this ratio has a value of $\sqrt[12]{1/2}$, and it is stored in memory location R of your calculator.
- (b) Take the length of the open string as 650 mm and use the values stored in R and E to calculate the theoretical distances of the frets from the bridge, and the corresponding Rule of 18 distances. You might do this by using an amended form of the program SCALE to store the two sets of distances in lists L1 and L2.
- (c) Compare the two sets of distances, and comment on the accuracy of the Rule of 18.

Question 6**Unit 9, Sec. 4**

The note produced by a vibrating string is specified by the frequency of the vibration. The frequency at which a string vibrates is inversely proportional to its length. This can be expressed symbolically as $f = k \times 1/L$, where f is the frequency, L is the string length and k is a constant. Thus, halving the length of the string will double the frequency of vibration.

The value of the constant k for a particular string depends on two factors: the weight (or gauge) of the string (measured as mass per metre), and the tension of the string (a measure of how tightly the string is stretched). If you measure the length of the string in metres, then $k = \frac{1}{2}\sqrt{T/M}$, where M is the mass per unit length of the string, measured in kilograms per metre, and T is the tension, measured in newtons. Thus the full relationship between frequency, length, tension and mass per metre is

$$f = \frac{\sqrt{T/M}}{2L}.$$

So, for a particular string of fixed length, the tension would need to be quadrupled in order to double the frequency.

- (a) Rearrange the above equation to make T the subject.
- (b) The middle C string of a given piano is 64 cm long and weighs 3.8 g.
- (i) What is its mass per unit length, in kilograms per metre?
 - (ii) To what tension, in newtons, must the string be stretched in order to produce a frequency of 256 Hz (middle C)?

Question 7

Unit 9, Sec. 4

Ron, an MU120 student and a keen long-distance cyclist, wants to develop a formula like Naismith's rule to estimate the times of cycle journeys.

He frequently cycles 10 km from A over a hill to H. Using an OS map, he marks intermediate positions B, C, D, E, F and G where the road crosses contour lines on the map, measures the distance of each point from the start and notes its height above sea level as in the table below.

- (a) Input these data as lists in your calculator and plot the points as a line graph. What implicit assumptions are made in this representation?
- (b) Ron uses Naismith's rule to work out how long the journey would take if he were walking. Naismith's rule gives the time t hours taken to walk a distance d km involving *climbing* a height h m as

$$t = \frac{d}{5} + \frac{h}{600}.$$

Ron would prefer to work in minutes (T) rather than hours (t). Write down an equation linking T and t . Multiply every term in Naismith's rule by 60 to obtain an equation $T = \dots$

- (c) Use this rule to estimate the time taken in minutes to complete each section of the journey. Then fill in columns (4)–(6) of the table.

(1) Point on journey	(2) Distance from start/km	(3) Height above sea level/m	(4) Distance from previous point/km	(5) Height above previous point/m	(6) Time taken from previous point/min	(7) Mean cycling time taken by Ron/min
A	0	100				
B	2	100				
C	4	120				
D	5	120				
E	6	160				
F	7	180				
G	8	160				
H	10	140				

- (d) How you would expect the times in column (6) to differ if Ron were on his bike, rather than walking.
- (e) According to Naismith's rule, a walker does not go faster when going downhill, Ron knows that on his cycle he travels faster downhill than on the flat. So Naismith's rule will need to be changed to accommodate this.

Ron decides that Naismith's rule

$$T = 12d + \frac{h}{10}$$

could be amended for cycling, to Ron's rule

$$T = Pd + \frac{h}{Q} - \frac{h'}{R},$$

where h' m is the height descended, and P , Q and R are constants which he needs to work out from timing his cycle journeys along the individual sections on his way from A to H. Ron completes the journey several times, recording the number of minutes for the individual

sections. The mean times (to the nearest minute) are 0, 12, 16, 6, 14, 10, 2 and 8, respectively. Enter these in column (7).

- (f) Consider the parts of the journey where there is no change in height, (in other words, where $h = 0$ and $h' = 0$). Write down the corresponding simplified version of Ron's rule. Manipulate this equation to make P the subject. Use appropriate data from the table to calculate the value of P .

Hence write down a full version of Ron's rule, without using P .

- (g) Now consider the uphill parts of the journey where $h' = 0$. Write down the corresponding simplified version of Ron's rule, and manipulate the equation to make Q the subject. Use appropriate data from the table to calculate the value of Q .

Write down a full version of Ron's rule, without using P or Q .

- (h) Use a similar method to calculate the value of R , and write down a full version of Ron's rule, without using P , Q or R .
- (i) Ron now sees that, because of the values of Q and R , he can simplify the rule at a stroke. There is no need to think of h as the height climbed and h' as the height descended. They can both be replaced by a new variable (say, w) which represents the vertical displacement in metres. Then $w = h - h'$. Write down the final simplified version of Ron's rule, and test it for various parts of the cycle journey.
- (j) Use Ron's rule to calculate the time taken for the cycle journey back from H to A. Why do you think there is something wrong with Ron's rule?

Question 8

Unit 9, Sec. 4

A Pythagorean triple consists of three whole numbers that will satisfy Pythagoras' theorem. One such triple is $[3, 4, 5]$ because $3^2 + 4^2 = 5^2$.

- (a) Complete these Pythagorean triples: $[9, 40, \dots]$, $[5, 12, \dots]$.
- (b) There are several formulas for generating such triples, one of which is $[n, \frac{1}{2}(n^2 - 1), \frac{1}{2}(n^2 + 1)]$.

This works when n is any positive odd number over 2. Use this formula and your calculator to find the triples generated when

- (i) $n = 5$, (ii) $n = 9$, (iii) $n = 17$, (iv) $n = 21$.

- (c) Work through the steps below to prove that this formula works. Let $A = n$, $B = \frac{1}{2}(n^2 - 1)$ and $C = \frac{1}{2}(n^2 + 1)$. If $[A, B, C]$ is a Pythagorean triple, then it will satisfy Pythagoras' formula

$$A^2 + B^2 = C^2.$$

- (i) Work out A^2 and B^2 , and then $A^2 + B^2$. Simplify this expression.
- (ii) Work out separately and simplify the value of C^2 .
- (iii) Show that the expression in (i) is the same as the expression in (ii) and thereby prove that $[n, \frac{1}{2}(n^2 - 1), \frac{1}{2}(n^2 + 1)]$ is a Pythagorean triple.

Unit 6

1

- (a) The features are:
- 168829 tumulus
 - 147844 reservoir
 - 136880 Madwoman's Stones
 - 150829 church with tower.

- (b) The grid references are:
- top of Crook Hill 183868
 - Druid's Stone 134874
 - Blue John Cavern 131831
 - railway station 181832.

Remember that six-figure grid references specify the south-western corner of the 100 m square containing the feature. Do not worry if your references differ slightly from these.

2

The squares of the National Grid have sides of length 100 km, or 100 000 m. At a scale of 1:50 000, the map will be a square with sides of length $100\,000/50\,000 = 2$ m.

3

- (a) Multiply the map distance in centimetres by 25 000 (the map scale is 1:25 000), then divide by 100 000 to get the ground distance in kilometres. Round to 1 decimal place.

Feature	Map distance/ cm	Ground distance/ km
Guide post	5.0	1.3
Cairn	3.3	0.8
Madwoman's Stones	5.0	1.3
Shooting cabin	5.7	1.4

- (b) Distance to guide post:

$$\begin{aligned} &\sqrt{(160 - 147)^2 + (873 - 874)^2} \\ &= 13.04 \text{ units (of 100 m)} \\ &= 1.304 \text{ km.} \end{aligned}$$

The previous calculator entry can be edited to produce the calculations for the next three locations.

Distance to cairn at grid reference 149882:

$$\begin{aligned} &\sqrt{(149 - 147)^2 + (882 - 874)^2} \\ &= 8.25 \text{ units} = 0.825 \text{ km.} \end{aligned}$$

Distance to Madwoman's Stones:

$$\begin{aligned} &\sqrt{(136 - 147)^2 + (880 - 874)^2} \\ &= 12.53 \text{ units} = 1.253 \text{ km.} \end{aligned}$$

Distance to shooting cabin:

$$\begin{aligned} &\sqrt{(142 - 147)^2 + (887 - 874)^2} \\ &= 13.93 \text{ units} = 1.393 \text{ km.} \end{aligned}$$

These calculations confirm the answers obtained in part (a).

4

In 1994, the difference between magnetic north and grid north was estimated to be decreasing by about $\frac{1}{2} \div 3 = \frac{1}{6}^\circ$ per year. If this rate remains constant, the difference will decrease by $10 \times \frac{1}{6} = \frac{10}{6} \simeq 1.7^\circ$ between 1994 and 2004. So, in 2004, the direction of magnetic north in the region covered by your OS map extract will be about $5 - 1.7 = 3.3^\circ$ west of grid north.

5

Looking back from Win Hill, the compass bearing of the walker's position is 180° greater than the bearing she measured. So the reverse bearing from Win Hill is $83 + 180 = 263^\circ$. Similarly, the reverse bearing from the chimney is $119 + 180 = 299^\circ$.

To convert to grid bearings, subtract 5° from each bearing to compensate for the difference between grid north and magnetic north. Using a protractor and your OS map extract, draw lines at grid bearings of $263 - 5 = 258^\circ$ and $299 - 5 = 294^\circ$ from Win Hill and the chimney, respectively. You should find that the lines cross near Mam Farm, where the footpath meets the old road from Castleton (grid reference 132839). This is where the walker is standing.

6

Drawing up a table is a useful way of collecting together the relevant details. The table lists approximate distances on the walk from one point to the next and the corresponding heights ascended, all found from the OS map extract. Naismith's rule has then been used to estimate the time taken to walk from one point to the next; these times are given in the final column of the table.

Route	Distance/ km	Height ascended/m	Time/ hours
Car park	0	0	0
Hollins Cross	2.0	97	0.56
Backtor Bridge	1.4	0	0.28
Beginning of Harden Clough	1.6	22	0.36
Car park	2.3	223	0.83
Total	7.3	342	2.03

Naismith's rule requires knowledge of the distance and the height ascended for each leg of the walk. The time for each leg can then be calculated using the formula

$$\text{time (hours)} = \frac{\text{distance (km)}}{5} + \frac{\text{height ascended (m)}}{600}$$

For example, on the first leg to Hollins Cross a walker ascends a height of about 97 m to reach Mam Tor (after that it is downhill to Hollins Cross), over a distance of 2 km. Naismith's rule gives the time for this leg as

$$\begin{aligned} \text{time (hours)} &= \frac{2}{5} + \frac{97}{600} \\ &\simeq 0.4 + 0.16 \\ &= 0.56, \text{ or about 34 minutes.} \end{aligned}$$

Using Naismith's rule on each stage of the walk and adding the results gives an estimate of just over 2 hours to cover 7.3 km.

You should get the same result if you use Naismith's rule on the entire walk with the total distance of 7.3 km and 342 m for that total ascent.

7

The cairn at 147874 is at a height of just over 440 m above sea level. The table gives estimates of the height, change in height, distance and gradient for various features relative to this cairn.

Feature	Height/ m	Change in height/ m	Distance/ m	Average gradient
Guide post	300	-150	1250	-0.1
Cairn	440	0	825	0
Madwoman's Stones	565	125	1250	0.1
Shooting cabin	400	-40	1425	-0.03

Note that distances are given to 3 significant figures here and so differ slightly from those given in Solution 3.

8

In the imperial system, 3 miles are equal to 3×1760 yards. This is $3 \times 1760 \times 3$ feet, or $3 \times 1760 \times 3 \times 12$ inches = 190 080 inches. So 1 inch on the map represents 190 080 inches on the ground. Hence the map scale is 1 : 190 080.

As 1 inch represents 3 miles, or $3 \times 1.61 = 4.83$ km, an area of 1 square inch (1 inch $\times$ 1 inch) on the map represents an area of $4.83 \times 4.83 = 23.3289$ km² on the ground.

A hectare is an area of 10 000 m², which is 0.01 km² (because 1 km² = 1000 000 m²). So a map area of 1 square inch represents an area of $23.3289/0.01 = 2332.89 = 2333$ ha (to the nearest hectare).

9

The basic fact to use in this question is that the area scaling factor is equal to the square of the map scale.

- (a) An area of 1 cm² on the map represents an area of 50 ha on the ground. To find the area scaling factor, find the number of square centimetres in 50 ha.

$$\begin{aligned} \text{Now } 1 \text{ ha} &= 10\,000 \text{ m}^2, \text{ and} \\ 1 \text{ m}^2 &= 100 \times 100 \text{ cm}^2 = 10\,000 \text{ cm}^2. \text{ So} \\ 50 \text{ ha} &= 50 \times 10\,000 \text{ m}^2 = 500\,000 \text{ m}^2, \\ \text{which is equivalent to} \\ 500\,000 \times 10\,000 &= 5\,000\,000\,000 \text{ cm}^2 \\ &= 5 \times 10^9 \text{ cm}^2. \end{aligned}$$

Therefore, the area scaling factor of the map is 5 000 000 000, or 5×10^9 . This is the square of the map scale, so take the square root:

$$\sqrt{5 \times 10^9} \simeq 7.0711 \times 10^4 = 70\,711.$$

Thus the map scale is about 1 : 70 711.

- (b) If the map scale were 1 : 100 000, then the area scaling factor would be (100 000)². Therefore 1 cm² on the map would represent (100 000)² = 10 000 000 000 = 10¹⁰ cm² on the ground.

Now 1 ha = 10 000 m², and since 1 m² = 10 000 cm², there must be $10\,000 \times 10\,000 = 100\,000\,000 = 10^8$ cm² in 1 ha.

Hence 10¹⁰ cm² on the ground is the same as 10¹⁰/10⁸ ha, or 10 000 000 000/100 000 000 ha = 100 ha. So 1 cm² on the map represents 100 ha at this scale.

Notice that reducing the map scale from 1 : 70 711 to 1 : 100 000 corresponds to reducing the scale by a factor of $\sqrt{2}$, since $70\,711 \times \sqrt{2} \simeq 70\,711 \times 1.41421 \simeq 100\,000$. The effect is to increase the area represented by 1 cm² on the map by a factor of 2, since $\sqrt{2} \times \sqrt{2} = 2$.

10

The first task is to collect the data from the map in order to draw the profile of the route. Two sets of data are required for each point: height in metres (from the contours), and distance from the start point (the cairn at 147874). The distances are measured in millimetres on the map, and then converted to the distances on the ground in metres by using the scale of the map: one millimetre on the map is the equivalent of 25 m on the ground.

Below are the data collected by one person. However, it is very difficult to be accurate on these sorts of distances, so do not be surprised if your data differ slightly.

Height/m	440	430	420	410	400
Map distance/mm	1	5	7.5	9	10
Ground distance/m	25	125	188	225	250

390	380	370	360	350	340	340	350	360
11	12	13	14.5	15	15.2	16	17	18
275	300	325	363	375	380	400	425	450

370	380	390	400	410	420	430	440	442
19	20	21	22	24	26	28.5	32	33
475	500	525	550	600	650	713	800	825

Now plot the profile of the route on your calculator. Choose distance in metres as the x variable, and enter these data into list L1. Then enter the heights into list L2.

Next an appropriate window setting is needed. The ' x ' data, the ground distances in metres, go from 0 to 825 m, and the ' y ' data, the heights, go from 340 to 442 m. So the window settings you might try are: 0 to 850, in steps of 100; and 330 to 450, in steps of 10. Set up a statistical plot, choosing a line graph of the data in L1 and L2.

Pressing GRAPH gives the profile, which should look something like that in Figure 1.

From this graph you can see that the gradient is close to zero at both cairns, and it is actually zero in the bed of the brook which is at a height of 340 m and at a distance of between 380 and 400 m from the first cairn.

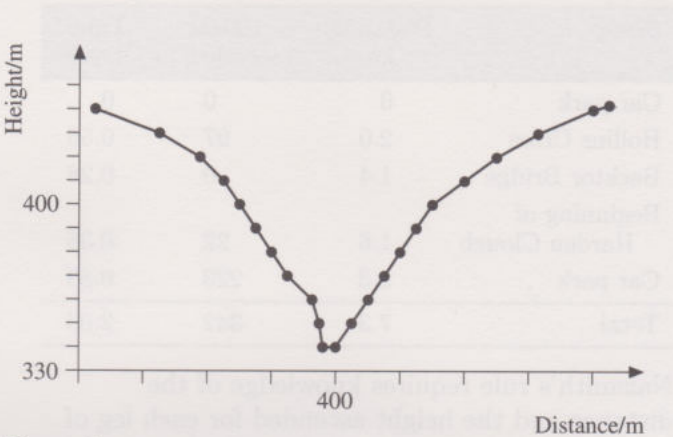


Figure 1

11

- (a) The grid reference of a point has six figures. The first three give the 'easting' of the point, and the last three give the 'northing'. So the data to be entered into lists L1 and L2 are:

Eastings (L1)	197	197	190	183	187	197
Northings (L2)	849	856	860	850	851	849

The easting values are all within the range 183 to 197, and the northings are between 849 and 860. A small extension at each end of these ranges will place the graph in the centre of the screen, so set the first six window values to 180, 200, 1, 845, 865, 1. Display the graph by selecting, from the STAT PLOT menu, a line graph, L1 and L2. Your graph representing the route of the walk should be similar to Figure 2.

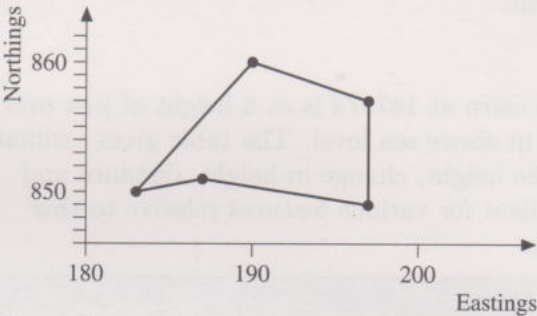


Figure 2

- (b) The method is explained in the *Calculator Book*, Section 6.2.

The table shows the values displayed in the calculator's lists at the end of the process.

L1	L2	L3	L4	L5
197.0	849.0	197.0	856.0	.7
197.0	856.0	190.0	860.0	.8
190.0	860.0	183.0	850.0	1.2
183.0	850.0	187.0	851.0	.4
187.0	851.0	197.0	849.0	1.0

The total distance of the walk is the sum of the values in L5—about 4.1 km.

Unit 7

1

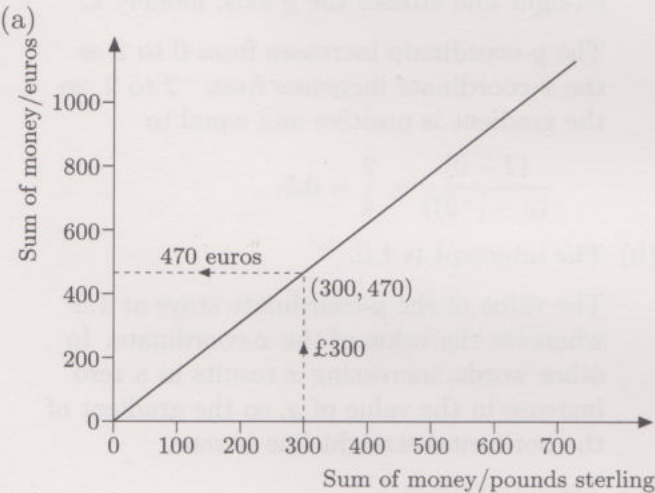


Figure 3

The graph is drawn so that it passes through the two fixed points (300, 470) and (0, 0).

- (b) From the graph, you should find that £400 are equivalent to about 630 euros, and that 750 euros are equivalent to about £480.
- (c) The graph is linear (that is, a straight line) and, because it goes through the origin, it shows a directly proportional relationship between euros and pounds.

2

- (a) The current bird populations are 5200 in sanctuary A and 6000 in sanctuary B. The population of sanctuary A is growing by 500 birds a year, and the population of B by 300 birds per year. So, the population of sanctuary A will be $5200 + 500 = 5700$ after the first year, $5200 + (500 \times 2) = 6200$ after the second year, $5200 + (500 \times 3) = 6700$ after the third, and so on. The bird population of sanctuary B grows in a similar way. So the general word formula for predicting bird populations is

bird population after a number of years
= current number of birds +

(average growth per year $\times$ number of years).

For sanctuary A, this gives

bird population
= $5200 + (500 \times \text{number of years})$,

and for sanctuary B,

bird population
= $6000 + (300 \times \text{number of years})$.

- (b) You can plot the growth of the bird populations as time increases on a graph where the vertical axis represents the number of birds and the horizontal axis represents the time in years, starting from the present.

The population growth in each sanctuary is represented by a straight line. The line for sanctuary A has an intercept of 5200 and a gradient of 500, and that for sanctuary B an intercept of 6000 and a gradient of 300 (see Figure 4).

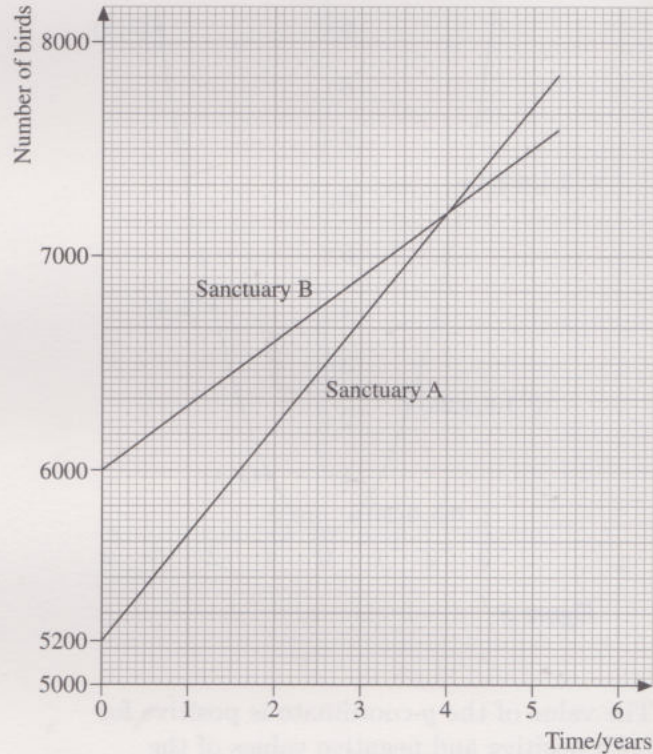


Figure 4

The graphs cross when the number of years is 4 and the estimated population of both sanctuaries is 7200. Therefore the population estimates predict that the populations will be equal after about 4 years.

You can check this result numerically. After 4 years the number of birds in sanctuary A will be $5200 + (500 \times 4) = 7200$. In sanctuary B the bird population will be $6000 + (300 \times 4) = 7200$, the same as in sanctuary A.

3

x-coordinate	y-coordinate	Coordinate pair
-2.5	6.25	(-2.5, 6.25)
-1.5	2.25	(-1.5, 2.25)
-0.5	0.25	(-0.5, 0.25)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

The graph drawn from these data is shown in Figure 5.

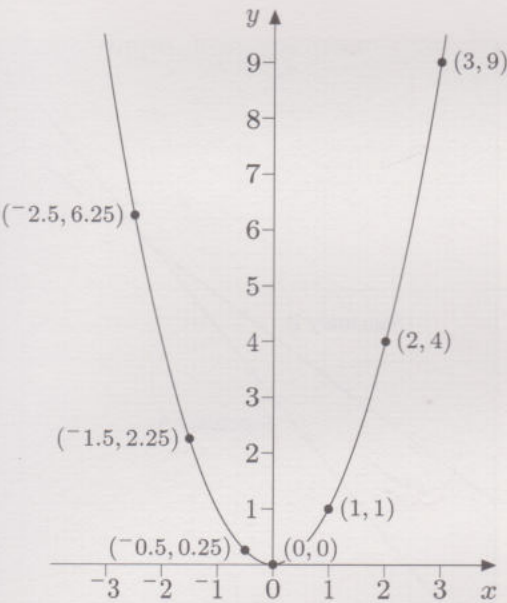


Figure 5

The value of the y -coordinate is positive for both positive and negative values of the x -coordinate (because the square of a negative number is always a positive number). For negative values of the x -coordinate, the slope or gradient of the graph is negative (sloping down from left to right), becoming less steep as the x -coordinate gets closer to zero. At the origin of the graph, $(0, 0)$, the slope is zero. For positive values of the x -coordinate, the slope is positive (sloping up from left to right), becoming steeper as the x -coordinate increases.

4

The intercept of a straight-line graph is the value of the y -coordinate when the x -coordinate is zero.

The gradient is the slope of the graph. It is defined as

$$\text{gradient} = \frac{\text{increase in } y\text{-coordinate}}{\text{increase in } x\text{-coordinate}}.$$

If the y -coordinate decreases as the x -coordinate increases, the change in y is taken as a *negative* increase.

- (a) The y -axis is the line where $x = 0$. Therefore the intercept is where this straight line crosses the y -axis, namely 1.

The y -coordinate increases from 0 to 2 as the x -coordinate increases from -2 to 2, so the gradient is positive and equal to

$$\frac{(2 - 0)}{(2 - (-2))} = \frac{2}{4} = 0.5.$$

- (b) The intercept is 1.5.

The value of the y -coordinate stays at 1.5 whatever the value of the x -coordinate. In other words, increasing x results in a zero increase in the value of y , so the gradient of the horizontal straight line is zero.

- (c) The intercept is 2.

This gives one suitable point to use in determining the gradient of the line, namely $(0, 2)$. Choose another point, say $(2, -1)$. As the x -coordinate increases from 0 to 2, the y -coordinate changes from 2 to -1, an 'increase' of -3. So the gradient of the line is $-\frac{3}{2} = -1.5$.

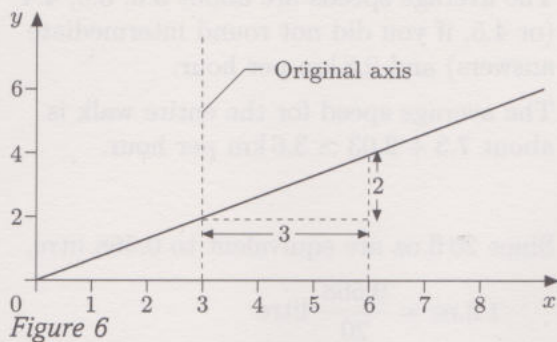
- (d) The straight line passes through the origin of the graph, so the intercept is zero.

The graph also passes through the point $(3, 2)$, so the gradient is $\frac{2}{3}$, or 0.67.

- (e) In this graph the axes meet not at the origin but at $(3, 0)$, so the intercept (the value of the y -coordinate when the x -coordinate is zero) cannot be read off directly.

The graph is redrawn in Figure 6, with the x -axis extended back to zero. You can see that the straight line passes through the origin, so the intercept is zero.

The gradient is $\frac{2}{3}$, as can be seen from either of the dotted triangles.



[The relationship between y and x on this graph is the same as that in (d). In fact, (d) and (e) are the same graph drawn on different axes.]

- (f) In this graph the x -axis starts at 5 and the y -axis starts at 2, so you must be careful about finding the intercept. The graph is redrawn below with the x -axis extended to zero, clearly showing that the intercept is 7.

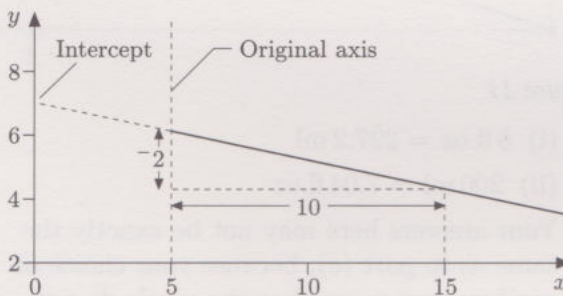


Figure 7

The x -coordinate increases from 5 to 15, and the y -coordinate decreases from 6 to 4. So the gradient is $\frac{-2}{10} = -0.2$.

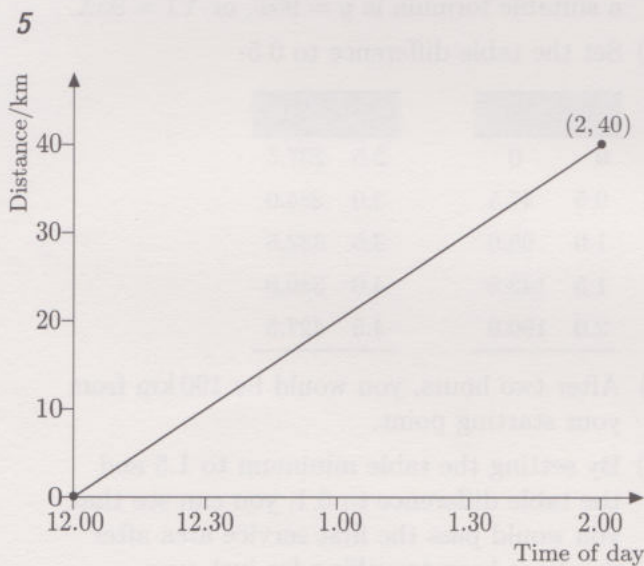


Figure 8

The distance-time graph is shown above. The straight line starts at the origin of the

graph—the origin represents the ferry's departure from the English port at 12 noon; it finishes at the point (2, 40), which represents the ferry's arrival at the French port at 2.00 p.m. (GMT).

- (a) The ferry has gone 15 km by 12.45 p.m., and 35 km by 1.45 p.m.
 (b) By 12.15 p.m. the ferry has gone 5 km from the English port, and by 1.10 p.m. it has gone about 23 km.
 (c) The gradient of the graph is $40/2 = 20$.
 (d) The ferry's average speed is 20 km per hour.

6

Measure distance from the port in nautical miles, and time after the radio message in minutes. Thus the origin of the distance-time graphs, (0, 0), represents the port at the moment the radio message was sent. Both vessels are assumed to have been travelling at constant speeds, 10 knots for the ship and 5 knots for the tug. Assume that they travelled directly towards each other. On the basis of these assumptions, the distance-time graphs for the two vessels would be as shown in Figure 9.

At time 0 the ship was 10 nautical miles from port, so the point (0, 10) represents the position of the ship when it sent the message. If it had continued to travel towards the port at a steady speed of 10 knots, it would have arrived there after 1 hour. Thus the ship's predicted motion is represented by the line joining (0, 10) and (60, 0).

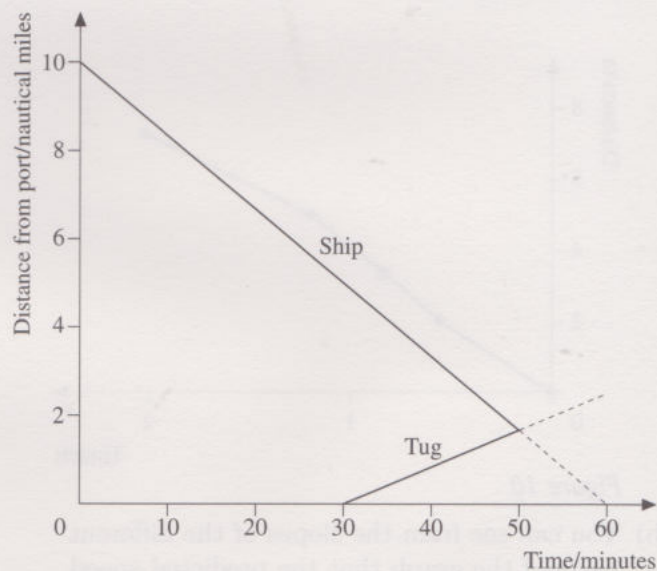


Figure 9

The tug started from port 30 minutes after the message was sent; this is represented by the point (30, 0). If it travelled at 5 knots for 30 minutes, it would have gone 2.5 nautical

miles in that time. Thus the tug’s predicted motion is represented by the line joining (30, 0) and (60, 2.5).

The distance–time graph predicts that the tug and the ship should meet (where the lines cross) about 50 minutes after the ship radioed. (There is no information about what happens when the ship and the tug do meet, so the broken lines indicate what their motion would have been if their speeds had remained unchanged.) Therefore the tug should start looking for the ship about 45 minutes after it received the radio message.

Since the tug would initially have been stationary in port it would have had to accelerate up to 5 knots. This means it might not have averaged as much as 5 knots over its voyage. So it might encounter the ship a bit later than 50 minutes after the radio message.

7

- (a) The vertical axis of the graph represents the distance in kilometres walked *from the start* (add together the distances for the relevant legs of the walk), and the horizontal axis represents the time in hours, measured *from the start* (add together the times for the relevant legs of the walk). So the start of the walk is represented by the point (0, 0) on the graph (Figure 10), and the finish, 2.03 hours later after walking 7.3 km, is represented by the point (2.03, 7.3). The intermediate points are plotted in a similar way.

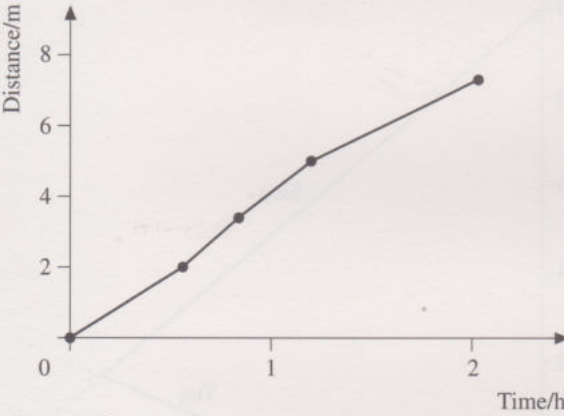


Figure 10

- (b) You can see from the slopes of the different parts of the graph that the predicted speed is greatest (the graph has the greatest slope) on the second leg: downhill between Hollins Cross and Backtor Bridge. The speed is least (the graph has the smallest slope) on the final (uphill) leg: along Harden Clough to the car park.

- (c) The average speeds are about 3.6, 5.0, 4.4 (or 4.5, if you did not round intermediate answers) and 2.8 km per hour.
- (d) The average speed for the entire walk is about $7.3 \div 2.03 \approx 3.6$ km per hour.

8

- (a) Since 20 fl.oz are equivalent to 0.568 litre,

$$\begin{aligned} 1 \text{ fl.oz} &= \frac{0.568}{20} \text{ litre} \\ &= 0.0284 \text{ litre, or } 28.4 \text{ ml.} \end{aligned}$$

- (b) Enter the function $Y1 = 28.4X$ and obtain a graph like that below.

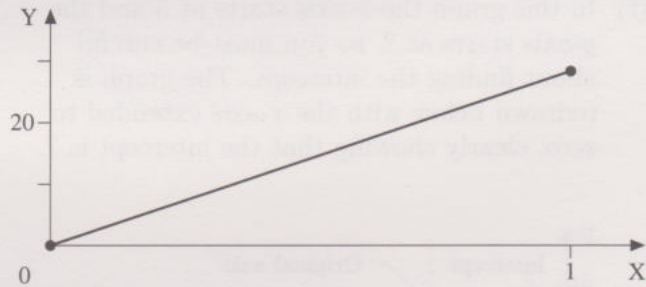


Figure 11

- (c) (i) 8 fl.oz = 227.2 ml
(ii) 200 ml = 7.04 fl.oz
- (d) Your answers here may not be exactly the same as in part (c), because your choice of window may mean that the pixels do not correspond exactly to 8 fl.oz or 200 ml.

9

- (a) With y for the distance travelled in kilometres and x for the time taken in hours, a suitable formula is $y = 95x$, or $Y1 = 95X$.
- (b) Set the table difference to 0.5.

X	Y1	X	Y1
0	0	2.5	237.5
0.5	47.5	3.0	285.0
1.0	95.0	3.5	332.5
1.5	142.5	4.0	380.0
2.0	190.0	4.5	427.5

- (c) After two hours, you would be 190 km from your starting point.
- (d) By setting the table minimum to 1.5 and the table difference to 0.1, you can see that you would pass the first service area after you have been travelling for just over 1.8 hours, or about 1 hour 50 min, and the second after you have been travelling for just over 2.3 hours, or about 2 hours 20 min.

10

To convert mass in pounds, P , to mass in stones, S , the formula is $S = P/14$.

Both the formulas $S = P/14$ and $K = 0.45P$ need to be converted into forms suitable for your calculator: these are $Y1 = X/14$ and $Y2 = .45X$, where $Y1$ is the number of stones, $Y2$ is the number of kilograms and X is the number of pounds. Since there are 7 lbs in a half-stone you want P (or X) to go up in intervals of 7. So set the table difference to 7, and obtain the following table:

X	Y1	Y2
0	0	0
7	.5	3.15
14	1	6.3
21	1.5	9.45
28	2	12.6
35	2.5	15.75
42	3	18.9

Unit 8

1

Here are some points to consider when you are manipulating positive and/or negative quantities.

- Numbers or symbols with no sign in front of them are positive.
- Decide whether the operation you are being asked to do is addition or subtraction, or multiplication or division.
- If it is addition or subtraction, remember that adding a negative number is the same as *subtracting* the corresponding positive number, but subtracting a negative number is the same as *adding* the corresponding positive number.
- If it is multiplication or division, then use the rule from *Preparatory Resource Book A*, Module 1:

If the two signs of the quantities to be multiplied or divided are the same, then the answer will be positive. If they are different, the answer will be negative.

- A minus sign outside a set of brackets changes all the signs inside the brackets.

Here are the answers:

- (a) 6 (b) -2 (c) $4a$
 (d) This expression is the same as $4 + 3 = 7$.

- (e) This is the same as $6 - 8 = -2$.
 (f) This is the same as $12 - 4 = 8$.
 (g) This is the same as $121p + 43p + 46p = 210p$.
 (h) -56
 (i) This is the same as $+3 \times +4 \times -9 = +12 \times -9 = -108$.
 (j) 2304 (k) -144 (l) -0.5 (m) $4q$
 (n) This is the same as $-4 + (+12) = 8$.
 (o) This is the same as $-3(-3) = 9$.

2

- (a) The first step in finding a formula is to decide which variables are to be represented by letters. It is tempting to think 'b for bananas', but is it the number of bananas, the price of bananas, the weight of bananas, or something different? You know that the price of bananas is 95p per kg and it is the number of kilograms that varies. So let n (or b or W or whatever) stand for the *number* of kilograms of bananas. Let c stand for the cost in pence of the bananas.

Now the cost of 1 kg of bananas is 95p, so the cost of n kg is $n \times 95p$.

Therefore

$$c = 95 \times n, \quad \text{or} \quad c = 95n.$$

- (b) For a carrier bag, 7p will be added to the cost of the purchase, giving a total cost of $95n + 7$ pence.

Note that the '7' stays as '7' irrespective of the weight of bananas purchased. (Assume only one bag is needed.)

Notice that this is the total cost in pence. If T is the total cost in £s, then

$$T = (95n + 7)/100.$$

- (c) The cost of 2.6 kg of bananas in a carrier bag is $\pounds(95 \times 2.6 + 7)/100$, which equals $\pounds2.54$.

3

A formula is needed that gives the relationship between the number of units used (denoted by, say, u) and the amount of the quarterly bill in £s (denoted by, say, b).

The cost of the calls alone will be the number of units multiplied by 2.4 p, but to get the answer in £s this must be divided by 100. So the cost of calls in £s = (number of units $\times$ 2.4)/100, or $0.024u$.

But the quarterly bill equals the cost of the calls plus the cost of the standing charge, so

$$b = 0.024u + 15.30.$$

4

By substituting each of the dimensions in cubits into the conversion formula provided, the dimensions of the ark, in metres, are found to have been 138 m by 23 m by 13.8 m.

This means that it was somewhat larger than a football pitch but still a bit small by the standards of a modern cruise ship. (The QE2 is 294 m long and 32 m broad.)

5

(a) When $q = 2$, then

$$p = 3 \times 2 + 2 = 8.$$

(b) When $q = 1.74$, then

$$p = 3 \times 1.74 + 2 = 7.22.$$

(c) When $q = -1.5$, then

$$p = 3 \times -1.5 + 2 = -2.5.$$

6

(a) When $x = 2$, then

$$2x^2 - 3 = 2 \times 4 - 3 = 5.$$

(b) When $q = \frac{4}{5}$, then

$$\begin{aligned} \frac{2}{3}(4 + 3q) &= \frac{2}{3}\left(4 + \frac{12}{5}\right) \\ &= \frac{2}{3}\left(\frac{20 + 12}{5}\right) \\ &= \frac{2}{3}\left(\frac{32}{5}\right) \\ &= \frac{64}{15}, \text{ or } 4\frac{4}{15}. \end{aligned}$$

7

(a) $5N$, $30z$, $40y$

(b) $-5y$, $22x$, $2p + 9p^2$, or $9p^2 + 2p$

8

In all cases, collect like terms.

(a) $4y + 4x$, or $4(y + x)$

(b) $5p + 2r$, or $2r + 5p$

(c) $12\clubsuit + 14\spadesuit$ (Note that symbols do not have to be letters.)

(d) This word formula can be written in symbols as $4k - 13k + s = -9k + s = s - 9k$ (it is usual to put the positive quantity first).

(e) $2A^2 + 3A^2 = 5A^2$

(f) The term ab is the same as ba because the order of symbols within a term does not matter. So the first and third terms in this expression are like terms and can be combined to give $1ab$, or just ab . Similarly, the second and fourth terms can be combined to give $4ab^2$. Therefore, the simplified expression is $ab + 4ab^2$, or $4ab^2 + ab$.

9 Remember that anything (number or symbol) outside the brackets multiplies everything inside the brackets.

(a) $6x - 2$ (b) $4k + 20$ (c) $2pq + p^2$

(d) $2ma - 2mb + 2mc$ (e) $2b^2 + 2b^3 - 2b^4$

(f) A minus sign before the term immediately outside the brackets multiplies everything by -1 , so every sign inside the brackets is changed:

$$-3x - 6y + 3z, \text{ or } 3z - 3x - 6y.$$

(g) $2r - 3r^2$

10

(a) Using the FOIL method from Unit 8:

$$\begin{aligned} (x + 2)(x + 5) &= x^2 + 5x + 2x + 10 \\ &= x^2 + 7x + 10. \end{aligned}$$

Alternatively, use the table method:

	x	5
x	x^2	$5x$
2	$2x$	10

(b) Using the FOIL method:

$$\begin{aligned} (2x - 1)(3x + 2) &= 6x^2 + 4x - 3x - 2 \\ &= 6x^2 + x - 2. \end{aligned}$$

Alternatively, use the table method:

	$3x$	2
$2x$	$6x^2$	$4x$
-1	$-3x$	-2

(c) Using the FOIL method:

$$\begin{aligned} (3 - y)(4 + y) &= 12 + 3y - 4y - y^2 \\ &= 12 - y - y^2. \end{aligned}$$

Alternatively, use the table method:

	4	y
3	12	$3y$
$-y$	$-4y$	$-y^2$

- (d) Using the FOIL method:

$$\begin{aligned}(5f - 3)(2f - 1) &= 10f^2 - 5f - 6f + 3 \\ &= 10f^2 - 11f + 3.\end{aligned}$$

Alternatively, use the table method:

	$2f$	-1
$5f$	$10f^2$	$-5f$
-3	$-6f$	3

- (e) Using the FOIL method:

$$(2 + x)(5 - y) = 10 - 2y + 5x - xy.$$

Alternatively, use the table method:

	5	$-y$
2	10	$-2y$
x	$5x$	$-xy$

- (f) Using the FOIL method:

$$(p + q)(x + y) = px + py + qx + qy.$$

11

- (a) Yes: $4 \times 4 = 16$.
- (b) Yes: the order of symbols may be changed.
- (c) No: p does not equal p^2 , nor does q^2 equal q .
- (d) No: as in part (c), powers do not match those in the original expression.
- (e) Yes: it multiplies out to the given expression.
- (f) Yes: 32 divided by 2 is 16, and q^2 divided by q is q .

12

- (a) $Y1(3) = 9.7$,
 $Y1(13.72) = 52.58$,
 $Y1(-101.5) = -408.3$
- (b) You will need to use different window settings for each of the x values, but you should be able to produce similar answers to those in part (a).

13

- (a) Step 1: multiply by 4.2 (to get $4.2q$)
 Step 2: subtract 1.7 (to get $4.2q - 1.7$).
- (b) To rearrange the formula, undo the steps in the reverse order by 'doing the same to both sides':
- Step 1: add 1.7 (to get $p + 1.7 = 4.2q$)
 Step 2: divide by 4.2
 (to get $(p + 1.7)/4.2 = q$).

- (c) From step 2 of (b),

$$q = (p + 1.7)/4.2.$$

14

- (a) Step 1: multiply by 12 (to get $12x$)
 Step 2: add 5 (to get $5 + 12x$).

To rearrange the formula, undo the steps in the reverse order by 'doing the same to both sides':

Step 1: subtract 5 (to get $y - 5 = 12x$)
 Step 2: divide by 12 (to get $(y - 5)/12 = x$).

- (b) Step 1: divide by 3 (to get $x/3$)
 Step 2: add 9 (to get $x/3 + 9$).

To rearrange the formula, undo the steps in the reverse order by 'doing the same to both sides':

Step 1: subtract 9 (to get $y - 9 = x/3$)
 Step 2: multiply by 3 (to get $3(y - 9) = x$).

- (c) Step 1: multiply by -2 (to get $-2x$)
 Step 2: add 12 (to get $12 - 2x$).

To rearrange the formula, undo the steps in the reverse order by 'doing the same to both sides':

Step 1: subtract 12 (to get $y - 12 = -2x$)
 Step 2: divide by -2 (to get $(y - 12)/-2 = x$,
 which is the same as
 $(12 - y)/2 = x$).

- (d) Step 1: cube x (to get x^3)
 Step 2: multiply by 8 (to get $8x^3$).

To rearrange the formula, undo the steps in the reverse order by 'doing the same to both sides':

Step 1: divide by 8 (to get the $y/8 = x^3$)
 Step 2: take the cube root (to get $\sqrt[3]{y/8} = x$,
 or $\frac{\sqrt[3]{y}}{2} = x$).

15

- (a) To get $2x + 5$, x is multiplied by 2 and then 5 is added, in that order.
- (b) To make x the subject, undo these operations in the reverse order, so subtract 5, then divide by 2.

Thus

$$2x + 5 - 5 = 8 - 5.$$

This gives

$$2x = 8 - 5 = 3.$$

Then divide both sides by 2:

$$x = 3 \div 2 = 1.5.$$

Therefore, the solution $x = 1.5$.

- (c) If $x = 1.5$,

$$2x + 5 = 2 \times 1.5 + 5 = 8.$$

So $x = 1.5$ is the correct solution.

16

- (a) $x = 1.5$ (b) $x = -1.5$

- (c) Before solving, simplify by collecting like terms, so $4x = 18$. Then $x = 4.5$.

- (d) Simplify to obtain $5x - 10 = 8$. Then $x = 3.6$.

17

Let Gretel's age be x years. So Hansel's age is $x + 6$ years. In two years' time, Hansel will be $x + 8$ years old and Gretel will be $x + 2$ years old. Hansel will then be three times as old as Gretel, therefore

$$x + 8 = 3(x + 2).$$

Expanding the brackets gives $x + 8 = 3x + 6$, hence $2 = 2x$, or $x = 1$. So Gretel is one year old.

(Check: Hansel is now seven; in two years' time, he will be nine and Gretel will be three.)

18

- (a) When $x = 2.25$, $y = 9.328$ (to 3 d.p.).
 (b) When $y = 10$, x can have three different values: 2.28, -0.66 and -4.62 (all correct to 2 d.p.).

19

- (a) Seventeen fuschias at $\pounds f$ each, plus $\pounds 2.36$ for postage and packing, gave a total bill of $\pounds 11.71$. So the relevant equation is

$$17f + 2.36 = 11.71.$$

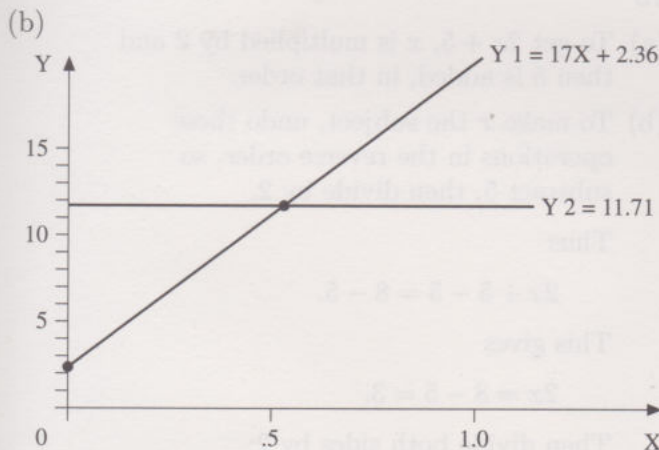


Figure 12

- (c) If $Y1 = 17X + 2.36$ and $Y2 = 11.71$, the two graphs will cross where $Y1 = Y2$, that is where $17X + 2.36 = 11.71$. The value of X at this point will be the solution of the equation. It is $X = 0.55$.

- (d) $Y1$ is the left-hand side of the equation in part (a) with f replaced by X , and $Y2$ is the right-hand side. So the intersection gives the solution of the equation in (a). Since $X = 0.55$, each fuschia cutting cost 55p.

- (e) $17f + 2.36 = 11.71$
 $17f = 11.71 - 2.36 = 9.35$
 $f = 9.35 \div 17 = 0.55.$

Algebraic solutions always give an exact answer; graphs do not.

20

- (a) The graph peaks at $y = 6.25$ and $x = 2.5$. So the firework reaches a maximum height of 6.25 m after 2.5 seconds.
 (b) The required function would be $y = 6$.
 (c) The graphs intersect twice, at about $x = 2$ and $x = 3$, indicating that the firework is at a height of 6 m after 2 seconds (on its way up) and after 3 seconds (on its way down).

Unit 5

1 (a)

- (i) $30^\circ = \frac{\pi}{6}$ radian (ii) $60^\circ = \frac{\pi}{3}$ radian
 (iii) $45^\circ = \frac{\pi}{4}$ radian (iv) $45^\circ = \frac{\pi}{4}$ radian
 (v) $60^\circ = \frac{\pi}{3}$ radian (vi) $30^\circ = \frac{\pi}{6}$ radian
 (vii) $45^\circ = \frac{\pi}{4}$ radian (viii) $30^\circ = \frac{\pi}{6}$ radian
 (ix) $60^\circ = \frac{\pi}{3}$ radian

(b)

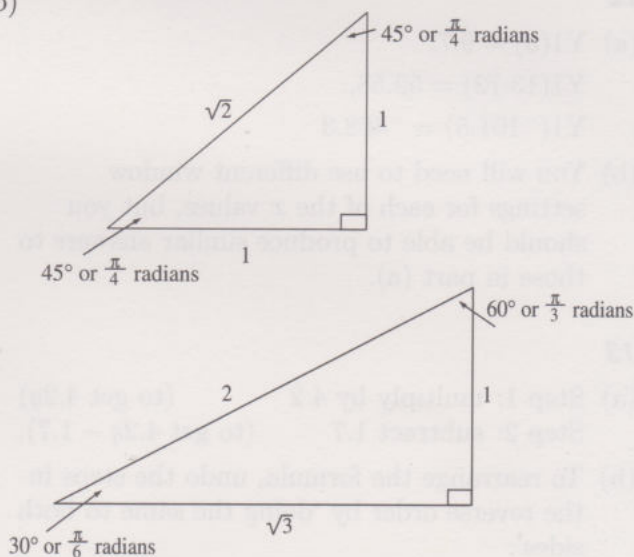


Figure 13

2

In TBLSET, you need to enter TblStart = 0 and $\Delta\text{Tbl} = \pi/6$.

Angle (X)/radian	Sine X	Cosine X
0	0	1
$\pi/6$	0.5	0.86603
$\pi/3$	0.86603	0.5
$\pi/2$	1	0
$2\pi/3$	0.86603	-0.5
$5\pi/6$	0.5	-0.86603
π	0	-1

In Question 1, $\sqrt{3}/2$ was used instead of 0.86603.

3

(a) *Answers in degrees*

(ii) $\sin 183^\circ = -0.052$ and $\sin 177^\circ = 0.052$.
Therefore, in general,
 $-\sin(180^\circ + x) = \sin(180^\circ - x)$.

(iii) $\cos 10^\circ = 0.985$ and $\cos -10^\circ = 0.985$.
Therefore, in general,
 $\cos x = \cos(-x)$.

(iv) $\cos 9^\circ = 0.988$ and $\cos 351^\circ = 0.988$.
Therefore, in general,
 $\cos x = \cos(360^\circ - x)$.

(v) $\sin 3^\circ = 0.052$ and $\cos 87^\circ = 0.052$.
Therefore, in general,
 $\sin x = \cos(90^\circ - x)$.

(vi) $\tan 5^\circ = 0.087$ and $\tan 185^\circ = 0.087$.
Therefore, in general,
 $\tan x = \tan(180^\circ + x)$.

(vii) $(\sin 37^\circ) \div (\cos 37^\circ) = \tan 37^\circ$
 $= 0.754$.

Therefore, in general,
 $(\sin x) \div (\cos x) = \tan x$.

(viii) $2(\sin 37^\circ)(\cos 37^\circ) = 0.961$ and
 $\sin 74^\circ = 0.961$.

Therefore, in general,
 $2 \sin x \cos x = \sin 2x$

(ix) $(\cos 19^\circ)^2 - (\sin 19^\circ)^2 = 0.788$ and
 $\cos 38^\circ = 0.788$.

Therefore, in general,
 $(\cos x)^2 - (\sin x)^2 = \cos 2x$.

(x) $(\cos 19^\circ)^2 + (\sin 19^\circ)^2 = 1$.
Therefore, in general,
 $(\cos x)^2 + (\sin x)^2 = 1$.

(b) *Answers in radians*

(ii) $-\sin(\pi + x) = \sin(\pi - x)$

(iii) $\cos x = \cos(-x)$

(iv) $\cos x = \cos(2\pi - x)$

(v) $\sin x = \cos(\pi/2 - x)$

(vi) $\tan x = \tan(\pi + x)$

(vii) $(\sin x)/(\cos x) = \tan x$

(viii) $2(\sin x)(\cos x) = \sin 2x$

(ix) $(\cos x)^2 - (\sin x)^2 = \cos 2x$

(x) $(\cos x)^2 + (\sin x)^2 = 1$.

4

(a) Both graphs are the same height in relation to the horizontal axis; in other words, they have the same amplitude. However, $y = \cos 2x$ has twice as many crests as $y = \cos x$; this means it has half the period (or wavelength) of $\cos x$.

(b) Both graphs have the same period, but the amplitude of $y = \sin 2x$ is twice the amplitude of $y = \sin x \cos x$.

(c) The graphs of Y1 and Y2 are identical, because $\sin x / \cos x = \tan x$.

(d) The graph of Y1 is the mirror image of that of Y2 (and, of course, vice versa).

(e) All values on both these graphs are positive. This is because any negative values, when squared, become positive.

The two graphs have the same shape but are shifted laterally by $\pi/2$ with respect to one another.

(f) The graph of Y3 is a straight line that has the value $y = 1$ for all values of x . This is because
 $Y3 = Y1 + Y2 = (\sin X)^2 + (\cos X)^2 = 1$.

5

- (a) Since 1/18 of the previous string length is marked off at each stage, the effective length of the string is reduced to 17/18 of its previous value, giving

Rule of 18 string-length ratio
= 17/18 = 0.9444 (to 4 d.p.).

However,

theoretical string-length ratio
= $\sqrt[12]{1/2}$ = 0.9439 (to 4 d.p.).

The values agree to 3 decimal places.

- (b) A suitable program is

```
PROGRAM:FRETS
:650 → L : 650 → M
:For (C, 1, 21)
:L → L1(C)
:M → L2(C)
:L * R → L
:M * E → M
:End
```

Notice how this is similar in structure to the program called SCALE.

By using this program, the following distances are obtained:

Fret number	Theoretical distance/mm	Rule of 18 distance /mm
0	650	650
1	614	614
2	579	580
3	547	548
4	516	517
5	487	488
6	460	461
7	434	436
8	409	411
9	386	389
10	365	367
11	344	347
12	325	327
13	307	309
14	290	292
15	273	276
16	258	260
17	243	246
18	230	232
19	217	219
20	205	207

- (c) As you saw in part (a), the Rule of 18 string-length ratio is slightly larger than the theoretical ratio, so it gives rise to slightly longer string lengths and, hence, greater fret distances. This effect becomes more marked at each stage—but even when the last fret is reached, the discrepancy is only about 2 mm (this corresponds to an interval of about one sixth of a semitone).

6

- (a) Multiply both sides by 2L:

$$2Lf = \sqrt{\frac{T}{M}}.$$

Square both sides:

$$4L^2f^2 = \frac{T}{M}.$$

Then multiply both sides by M:

$$T = 4L^2Mf^2.$$

- (b) (i) The mass of the string is 0.0038 kg, and its length is 0.64 m. The mass per unit length is thus
(0.0038/0.64) kg/m = 0.0059375 kg/m.
(ii) Evaluate the expression for T in part (a), with L = 0.64, M = 0.0059375 and f = 256. This gives the required tension, to the nearest newton, as 638 newtons.

Now, a 1 kg weight, hanging on a string, produces a tension of nearly 10 newtons. So the piano string is stretched as tightly as if it were carrying a weight of over 60 kg. Since a piano has a total of about 230 strings, each of which will be under similar tension, the total force on the frame of the piano will be comparable to a weight of between 10 and 20 tonnes. To withstand these enormous forces, the frame is constructed of heavy steel, with considerable bracing.

7

- (a) The graph may look something like Figure 14, depending on the window settings chosen.

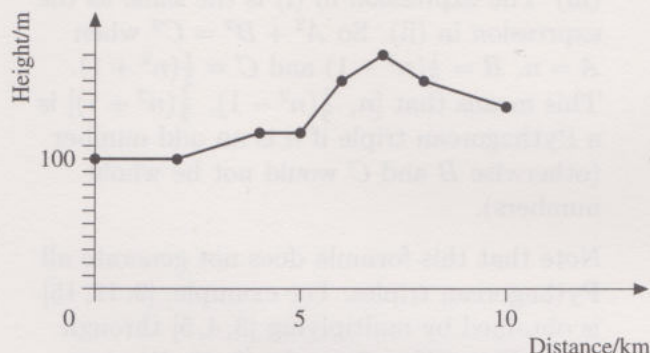


Figure 14

The implicit assumption in this graph is that the gradients between pairs of consecutive intermediate positions are constants.

- (b) To convert between time in minutes, T , and time in hours, t , the relevant equation is

$$T = 60t.$$

Now, according to Naismith's rule,

$$t = \frac{d}{5} + \frac{h}{600}.$$

Multiplying through by 60 gives

$$60t = \frac{60d}{5} + \frac{60h}{600}.$$

Substituting T for $60t$ gives

$$T = \frac{60d}{5} + \frac{60h}{600},$$

and simplifying each term produces

$$T = 12d + \frac{h}{10}.$$

- (c) See table below.

(1) Point on journey	(2) Distance from start/km	(3) Height above sea level/m	(4) Distance from previous point/km	(5) Height above previous point/m	(6) Time taken from previous point/min	(7) Mean cycling time taken by Ron/min
A	0	100	0	0	0	0
B	2	100	2	0	24	12
C	4	120	2	20	26	16
D	5	120	1	0	12	6
E	6	160	1	40	16	14
F	7	180	1	20	14	10
G	8	160	1	-20	12	2
H	10	140	2	-20	24	8

- (d) You would expect the times to be shorter, as Ron cycles faster than he walks, especially on the downhill sections.
- (e) See table below.
- (f) Where $h = 0$ and $h' = 0$, Ron's rule becomes

$$T = Pd.$$

So

$$P = T/d.$$

For the part of the journey from A to B (for which $h = 0$ and $h' = 0$), $d = 2$ and $T = 12$, hence $P = 6$.

The same result is obtained using data for the part of the journey from C to D.

Then the full version of Ron's rule is

$$T = 6d + \frac{h}{Q} - \frac{h'}{R}.$$

- (g) Where $h' = 0$, Ron's rule becomes

$$T = Pd + \frac{h}{Q}.$$

So, as $P = 6$,

$$Q = h/(T - 6d).$$

For the part of the journey from B to C (for which h increases but $h' = 0$), $h = 20$, $T = 16$ and $d = 2$. So substituting these values in the above equation gives $Q = 5$.

The same result is obtained using data for other parts of the journey where h increases but $h' = 0$.

When the values for P and Q are substituted into the equation for Ron's rule, the rule can be written as

$$T = 6d + \frac{h}{5} - \frac{h'}{R}.$$

- (h) From Ron's rule, if $h = 0$, then

$$T = Pd - \frac{h'}{R}.$$

But $P = 6$, so

$$R = -h'/(T - 6d).$$

For the part of the journey from F to G (for which $h = 0$), $h' = 20$, $T = 2$ and $d = 1$, so $R = 5$.

Then Ron's rule can be written as

$$T = 6d + \frac{h}{5} - \frac{h'}{5}.$$

- (i) Ron's rule:

$$T = 6d + (h - h')/5$$

becomes

$$T = 6d + \frac{w}{5}.$$

- (j) For the journey from H back to A, $d = 10$ and $w = -40$, so substituting into Ron's rule from part (i) gives

$$\begin{aligned} T &= 6 \times 10 - 40/5 \\ &= 60 - 8 \\ &= 52. \end{aligned}$$

Therefore, the time calculated for the journey is 52 minutes.

However, going from E to D the gradient is very steep, and Ron's rule predicts a time of -2 minutes, which is impossible. In practice, a cyclist would brake on such gradients. So the rule would not apply to such sharp inclines.

8

- (a) $[9, 40, 41]$, $[5, 12, 13]$.

- (b) When $n = 5$, the triple is $[5, 12, 13]$.

When $n = 9$, the triple is $[9, 40, 41]$.

When $n = 17$, the triple is $[17, 144, 145]$.

When $n = 21$, the triple is $[21, 220, 221]$.

- (c) (i) $A^2 = n^2$

$$\begin{aligned} B^2 &= \left[\frac{1}{2}(n^2 - 1)\right]^2 = \frac{1}{4}(n^2 - 1)^2 \\ &= \frac{1}{4}(n^2 - 1)(n^2 - 1) \\ &= \frac{1}{4}(n^2 \times n^2 - n^2 - n^2 + 1) \\ &= \frac{1}{4}(n^4 - 2n^2 + 1) \end{aligned}$$

$$\begin{aligned} A^2 + B^2 &= n^2 + \frac{1}{4}(n^4 - 2n^2 + 1) \\ &= n^2 + \frac{1}{4}n^4 - \frac{1}{2}n^2 + \frac{1}{4} \\ &= \frac{1}{4}n^4 + \frac{1}{2}n^2 + \frac{1}{4}. \end{aligned}$$

$$\begin{aligned} \text{(ii) } C^2 &= \left[\frac{1}{2}(n^2 + 1)\right]^2 = \frac{1}{4}(n^2 + 1)^2 \\ &= \frac{1}{4}(n^2 + 1)(n^2 + 1) \\ &= \frac{1}{4}(n^4 + 2n^2 + 1) \\ &= \frac{1}{4}n^4 + \frac{1}{2}n^2 + \frac{1}{4}. \end{aligned}$$

(iii) The expression in (i) is the same as the expression in (ii). So $A^2 + B^2 = C^2$ when $A = n$, $B = \frac{1}{2}(n^2 - 1)$ and $C = \frac{1}{2}(n^2 + 1)$. This means that $[n, \frac{1}{2}(n^2 - 1), \frac{1}{2}(n^2 + 1)]$ is a Pythagorean triple if n is an odd number (otherwise B and C would not be whole numbers).

Note that this formula does not generate all Pythagorean triples. For example, $[9, 12, 15]$ is obtained by multiplying $[3, 4, 5]$ through by 3. This differs from the $[9, 40, 41]$ given by the formula. The formula in this question always produces triples where the second and third values are consecutive numbers. This is because if you add 1 to $\frac{1}{2}(n^2 - 1)$ you get $\frac{1}{2}(n^2 + 1)$:

$$\begin{aligned} \frac{1}{2}(n^2 - 1) + 1 &= \frac{1}{2}n^2 - \frac{1}{2} + 1 \\ &= \frac{1}{2}n^2 + \frac{1}{2} \\ &= \frac{1}{2}(n^2 + 1). \end{aligned}$$

MU120 Open Mathematics

Resource Book A Units 1–5

Resource Book B Units 6–9

Resource Book C Units 10–13

Resource Book D Units 14–16

